Argomento 6

- Lezione 9
- Lezione 10

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Equazione di Helmholtz

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = \nabla\times\mathbf{J}_{mi} + j\omega\mu\,\mathbf{J}_i - \frac{\nabla\nabla\cdot\mathbf{J}_i}{j\omega\varepsilon_c}$$

$$k^2 = \omega^2\mu\,\varepsilon_c$$

La classe di soluzioni fornita dal-l'eq. di Helmholtz è più ampia di quella fornita dal sistema di eq. di Maxwell (operazione di derivazione)

$$\nabla\cdot\mathbf{E} = -\frac{\nabla\cdot\mathbf{J}_i}{j\omega\varepsilon_c}$$

quindi tra tutte le soluzioni dell'equazione di Helmholtz scegliamo quelle che soddisfano anche la

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = 0 \qquad k^2 = \omega^2\mu\,\varepsilon_c$$

$$\nabla\cdot\mathbf{E} = 0$$

OMOGENEA

Soluzione equazione di H.

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0 \qquad k^2 = \omega^2 \mu \, \varepsilon_c$$

$$\mathbf{E}(x,y,z) = \mathbf{E}_0 X(x) Y(y) Z(z)$$

Metodo di soluzione per separazione delle variabili

vettore complesso *funzioni scalari complesse*

$$\boxed{\mathbf{E}(x, y, z) = \mathbf{E}_0 e^{-j\left(k_x x + k_y y + k_z z\right)}}$$

determina la polarizzazione del campo elettrico

determina la propagazione cioè la dipendenza dalle coordinate

$$k_x{}^2 + k_y{}^2 + k_z{}^2 = k^2$$

Condizione di separabilità

$$k_x x + k_y y + k_z z = \mathbf{k} \cdot \mathbf{r}$$

k=vettore di propagazione

$$\mathbf{E}(x, y, z) = \mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{-j\,\mathbf{k}\cdot\mathbf{r}}$$

$$\mathbf{k} = \boldsymbol{\beta} - j\boldsymbol{\alpha}$$

α=vettore di attenuazione

β=vettore di fase

$$\boxed{\mathbf{E}(x, y, z) = \mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{-\boldsymbol{\alpha}\cdot\mathbf{r}} e^{-j\,\boldsymbol{\beta}\cdot\mathbf{r}}}$$

sol. dell'eq. vettoriale di H. omogenea

Determinazione di k (come modulo)

$$\mathbf{E}(x,y,z)=\mathbf{E}(\mathbf{r})=\mathbf{E}_0 e^{-j\,\mathbf{k}\cdot\mathbf{r}}$$

→ *Rappresenta un'onda piana*

$$k^2=\omega^2\mu\varepsilon_c=\omega^2\mu\varepsilon-j\omega\mu\sigma$$

$$k=\sqrt{k^2}=\omega\sqrt{\mu\varepsilon_c}=\omega\sqrt{\frac{|\varepsilon_c\mu|+\mathrm{Re}(\varepsilon_c\mu)}{2}}-j\omega\sqrt{\frac{|\varepsilon_c\mu|-\mathrm{Re}(\varepsilon_c\mu)}{2}}$$

$$k=k_R-jk_J$$

$$\Rightarrow$$

$$k_R=\omega\sqrt{\frac{|\varepsilon_c\mu|+\mathrm{Re}(\varepsilon_c\mu)}{2}}$$

$$k_J=\omega\sqrt{\frac{|\varepsilon_c\mu|-\mathrm{Re}(\varepsilon_c\mu)}{2}}$$

Caratteristiche di propagazione delle onde piane

$$\boxed{\mathbf{E}(x,y,z)=\mathbf{E}(\mathbf{r})=\mathbf{E}_0 e^{-j\,\mathbf{k}\cdot\mathbf{r}}}$$

$$\mathbf{k}=\boldsymbol{\beta}-j\boldsymbol{\alpha}$$

$$\boxed{\mathbf{E}(x,y,z)=\mathbf{E}(\mathbf{r})=\mathbf{E}_0 e^{-\boldsymbol{\alpha}\cdot\mathbf{r}} e^{-j\,\boldsymbol{\beta}\cdot\mathbf{r}}}$$

Consideriamo un mezzo <u>*L-S-O-I-nonD->*</u>*ε, μ reali positivi*

$$k_x^{\ 2}+k_y^{\ 2}+k_z^{\ 2}=k^2=\mathbf{k}\cdot\mathbf{k}$$

$$k^2=\mathbf{k}\cdot\mathbf{k}=\omega^2\mu\varepsilon_c=\omega^2\mu\varepsilon-j\omega\mu\sigma$$

$$(\boldsymbol{\beta}-j\boldsymbol{\alpha})\cdot(\boldsymbol{\beta}-j\boldsymbol{\alpha})=\beta^2-\alpha^2-2j\boldsymbol{\beta}\cdot\boldsymbol{\alpha}=k^2=\omega^2\mu\varepsilon-j\omega\mu\sigma$$

$$\beta^2-\alpha^2=\omega^2\mu\varepsilon$$ *parte reale* $\longrightarrow$ $\beta>\alpha$

$$\boldsymbol{\beta}\cdot\boldsymbol{\alpha}=\frac{\omega\mu\sigma}{2}$$ *parte immaginaria*

Casi particolari

1) Mezzo privo di conducibilità (senza perdite) $\sigma = 0$

$$k^2 = \omega^2 \mu\varepsilon$$

$$\boldsymbol{\beta} \cdot \boldsymbol{\alpha} = 0$$

a) verificata per $\alpha = 0$ ***Onda piana e uniforme***

$$\mathbf{k} = \boldsymbol{\beta} = \beta\, \boldsymbol{\beta}_0 = k\, \boldsymbol{\beta}_0 = \omega\sqrt{\mu\varepsilon}\, \boldsymbol{\beta}_0$$

k=reale

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}$$

velocità della luce nel mezzo

b) verificata per $\alpha \perp \beta$ ***Onda piana non uniforme***

attenuata in direzione perpendicolare alla direzione di propagazione

Onda evanescente

k=reale

$$\beta = \sqrt{\omega^2\mu\varepsilon + \alpha^2} = \omega\sqrt{\mu\varepsilon}\sqrt{1 + \frac{\alpha^2}{\omega^2\mu\varepsilon}} > \omega\sqrt{\mu\varepsilon}$$

$$u = \frac{1}{\sqrt{\mu\varepsilon}} \frac{1}{\sqrt{1 + \frac{\alpha^2}{\omega^2\mu\varepsilon}}} < \frac{1}{\sqrt{\mu\varepsilon}}$$

Onda piana non uniforme

$$u_r = \frac{1}{\sqrt{\mu\varepsilon}} \frac{1}{\sqrt{1+\frac{\alpha^2}{\omega^2\mu\varepsilon}}} \frac{1}{\cos\theta}$$

$$se \quad \theta_0 = \arccos\frac{1}{\sqrt{1+\frac{\alpha^2}{\omega^2\mu\varepsilon}}}$$

$$\Downarrow$$

$$u_r = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$per \quad \theta < \theta_0 \Rightarrow u_r < \frac{1}{\sqrt{\mu\varepsilon}}$$

onda lenta

$$per \quad \theta > \theta_0 \Rightarrow u_r > \frac{1}{\sqrt{\mu\varepsilon}}$$

onda veloce

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Esempio onda piana e uniforme

Calcolo parametri di base:

Un'onda piana uniforme con una frequenza di 3 GHz si propaga in un mezzo senza superfici di discontinuità con ε_r=7 e μ_r=3. Calcolare la velocità di fase e la lunghezza d'onda:

$$u = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\mu_r \varepsilon_r}} = \frac{3\,10^8}{\sqrt{(7)(3)}} = 6.55\,10^7 \;\; m/s$$

$$\sqrt{21} = 4.58 < della \;\; velocità \; della \; luce$$

$$\lambda = \frac{u}{f} \quad \Rightarrow \qquad \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{\mu\varepsilon}} = \frac{2\pi u}{2\pi f} = \frac{u}{f} = \frac{6.55 \;\; 10^7}{3 \;\; 10^9} \approx 2 \;\; 10^{-2} \;\; m$$

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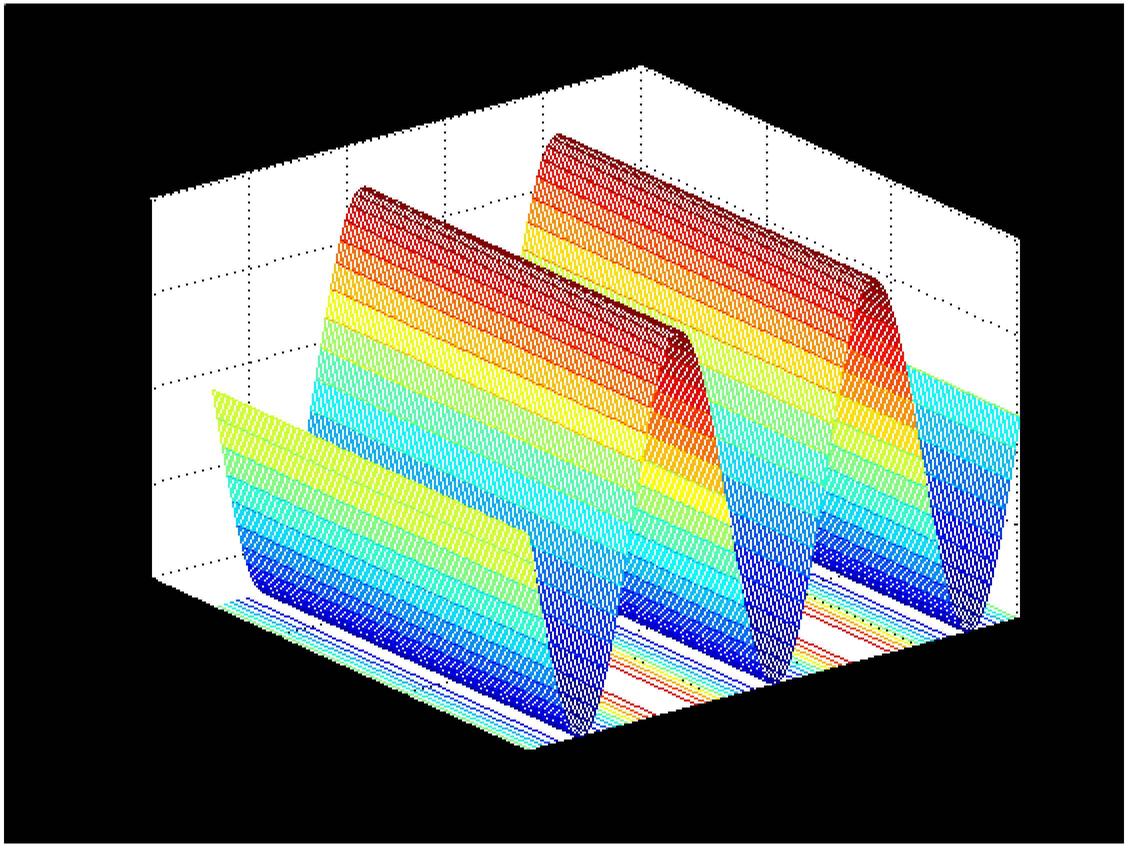
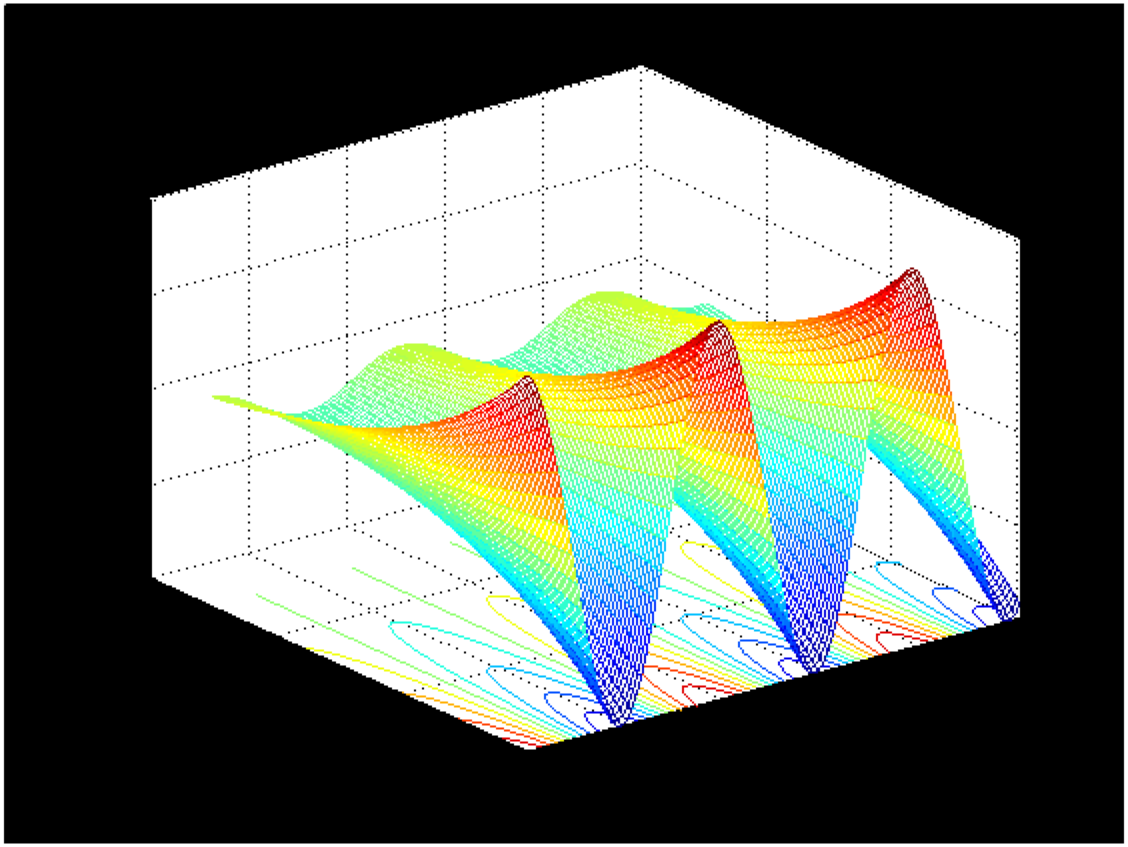
Onda piana e uniforme



Onda evanescente



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2) Mezzo dissipativo (con perdite) $\sigma \neq 0$

$$k^2 = \omega^2 \mu \varepsilon_c = \omega^2 \mu \varepsilon - j\omega\mu\sigma = \omega^2 \mu \varepsilon \left[1 - j\left(\frac{\sigma}{\omega\varepsilon}\right)\right]$$

k=complesso

β e α *entrambi non nulli*

Caso particolare $\alpha // \beta$ ***Onda piana uniforme e attenuata***

$$\boldsymbol{\beta} = \beta\,\boldsymbol{\beta}_0 \qquad \boldsymbol{\alpha} = \alpha\,\boldsymbol{\alpha}_0 \quad \Rightarrow \quad \mathbf{k} = (\beta - j\,\alpha)\boldsymbol{\beta}_0 = k\,\boldsymbol{\beta}_0$$

Onda che si propaga nella direzione di β *con velocità di fase* $u=\omega/\beta$*, lunghezza d'onda* $\lambda=2\pi/\beta$ *ed un fattore esponenziale di attenuazione.*

3) Mezzo dissipativo e dispersivo (con perdite) $\sigma \neq 0$ ε *complessa*

$$\varepsilon'' \neq 0$$

$$k^2 = \omega^2 \mu \varepsilon_c = \omega^2 \mu \varepsilon - j\omega\mu\sigma = \omega^2 \mu \varepsilon_0 \left(\varepsilon' - j\varepsilon''\right) - j\omega\mu\sigma =$$

$$= \omega^2 \mu \varepsilon_0 \varepsilon' \left(1 - j\frac{\varepsilon''}{\varepsilon'} - j\frac{\sigma}{\omega \varepsilon_0 \varepsilon'}\right)$$

Perdite legate alla corrente di spostamento

Perdite legate alla corrente di conduzione

k=complesso

$$k = k_R - jk_J$$

$$k_R = \omega\sqrt{\frac{|\varepsilon_c \mu| + \mathrm{Re}(\varepsilon_c \mu)}{2}}$$

$$k_J = \omega\sqrt{\frac{|\varepsilon_c \mu| - \mathrm{Re}(\varepsilon_c \mu)}{2}}$$

$$\varepsilon_c = \varepsilon_0\left(\varepsilon' - j\varepsilon''\right) - j\frac{\sigma}{\omega} = \varepsilon_0 \varepsilon' - j\left(\varepsilon_0 \varepsilon'' + \frac{\sigma}{\omega}\right)$$

Caso particolare $\alpha // \beta$ ***Onda piana uniforme e attenuata***

Caratteristiche di polarizzazione delle onde piane

L'operatore ∇ *opera sulla funzione* $\mathbf{E}(x,y,z)=\mathbf{E}(\mathbf{r})=\mathbf{E}_0 e^{-j\,\mathbf{k}\cdot\mathbf{r}}$ *come* $-j\,\mathbf{k}$

$$e^{-j\,\mathbf{k}\cdot\mathbf{r}}=e^{-j\left(k_\mathbf{x}x+k_\mathbf{y}y+k_\mathbf{z}z\right)}$$

$$\nabla[\ ]=\left(\mathbf{x}_0\frac{\partial}{\partial x}+\mathbf{y}_0\frac{\partial}{\partial y}+\mathbf{z}_0\frac{\partial}{\partial z}\right)[\ ]=-j\left(\mathbf{x}_0k_x+\mathbf{y}_0k_y+\mathbf{z}_0k_z\right)[\ ]=-j\mathbf{k}[\ ]$$

$$\nabla^2\mathbf{E}+k^2\mathbf{E}=\nabla\cdot\nabla\ \mathbf{E}+k^2\mathbf{E}=\left(\nabla\cdot\nabla+k^2\right)\mathbf{E}=\mathbf{0}$$

$$(-j\mathbf{k})\cdot(-j\mathbf{k})+k^2=-\mathbf{k}\cdot\mathbf{k}+k^2=0\Rightarrow\mathbf{k}\cdot\mathbf{k}=k^2$$

condizione di separabilità

$$\nabla\cdot\mathbf{E}=-j\mathbf{k}\cdot\mathbf{E}$$
$$\nabla\times\mathbf{E}=-j\mathbf{k}\times\mathbf{E}$$

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$$\mathbf{E}(x, y, z) = \mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{-j\,\mathbf{k}\cdot\mathbf{r}}$$

La funzione d'onda è stata ottenuta come soluzione dell'eq. vettoriale di Helmholtz

$$\nabla \cdot \mathbf{E} = 0 \Rightarrow \mathbf{k} \cdot \mathbf{E} = 0 \quad \Longrightarrow \quad \mathbf{k} \cdot \mathbf{E_0} = 0$$

$$\mathbf{E_0} = \mathbf{E}_{0R} + j\mathbf{E}_{0J}$$

$$(\boldsymbol{\beta} - j\boldsymbol{\alpha}) \cdot (\mathbf{E}_{0R} + j\mathbf{E}_{0J}) = 0$$

$$\boldsymbol{\beta} \cdot \mathbf{E}_{0R} + \boldsymbol{\alpha} \cdot \mathbf{E}_{0J} = 0$$

$$\boldsymbol{\beta} \cdot \mathbf{E}_{0J} - \boldsymbol{\alpha} \cdot \mathbf{E}_{0R} = 0$$

Trovato il campo elettrico, quello magnetico si può ricavare dalla:

$$\nabla \times \mathbf{E} = -j\omega\mu\, \mathbf{H}$$

$$-j\mathbf{k} \times \mathbf{E} = -j\omega\mu\, \mathbf{H}$$

$$\mathbf{H} = \frac{\mathbf{k} \times \mathbf{E}}{\omega\mu} = \frac{\mathbf{k} \times \mathbf{E}_0}{\omega\mu} e^{-j\mathbf{k}\cdot\mathbf{r}} = \mathbf{H}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}$$

funzione d'onda dello stesso tipo di quella di **E**

$$\mathbf{H}_0 = \frac{\mathbf{k} \times \mathbf{E}_0}{\omega\mu}$$

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Casi particolari

1) Mezzo non dispersivo e non dissipativo ε, μ *reali e positive*; $\sigma = 0$

$\alpha = 0$ ***Onda piana e uniforme*** *(non attenuata)*

$\mathbf{k} = \boldsymbol{\beta}$

$$\boldsymbol{\beta} \cdot \mathbf{E}_{0R} + \boldsymbol{\alpha} \cdot \mathbf{E}_{0J} = 0 \qquad \Rightarrow \qquad \boldsymbol{\beta} \cdot \mathbf{E}_{0R} = 0$$

$$\boldsymbol{\beta} \cdot \mathbf{E}_{0J} - \boldsymbol{\alpha} \cdot \mathbf{E}_{0R} = 0 \qquad \qquad \boldsymbol{\beta} \cdot \mathbf{E}_{0J} = 0$$

$$\mathbf{H}_0 = \mathbf{H}_{0R} + j\mathbf{H}_{0J} = \frac{\boldsymbol{\beta} \times \mathbf{E}_0}{\omega\mu} = \frac{\boldsymbol{\beta} \times \left(\mathbf{E}_{0R} + j\mathbf{E}_{0J}\right)}{\omega\mu}$$

$$\mathbf{H}_{0R} = \frac{\boldsymbol{\beta} \times \mathbf{E}_{0R}}{\omega\mu} \qquad \mathbf{E}_{0R} = -\frac{\boldsymbol{\beta} \times \mathbf{H}_{0R}}{\omega\varepsilon}$$

$$\mathbf{H}_{0J} = \frac{\boldsymbol{\beta} \times \mathbf{E}_{0J}}{\omega\mu} \qquad \mathbf{E}_{0J} = -\frac{\boldsymbol{\beta} \times \mathbf{H}_{0J}}{\omega\varepsilon}$$

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Onda piana e uniforme *(non attenuata)*

la costante di propagazione coincidente con β è ortogonale a E_{0R} e a E_{0J}, H_{0R}, H_{0J} sia il vettore campo elettrico E che quello campo magnetico H non hanno componenti nella direzione di propagazione

Onda TEM (TrasversoElettroMagnetica)

1)

$$\boldsymbol{\beta} \cdot \mathbf{E}_{0R} = 0$$

$$\mathbf{H}_{0R} = \frac{\boldsymbol{\beta} \times \mathbf{E}_{0R}}{\omega\mu}$$

2)

$$\boldsymbol{\beta} \cdot \mathbf{E}_{0J} = 0$$

$$\mathbf{H}_{0J} = \frac{\boldsymbol{\beta} \times \mathbf{E}_{0J}}{\omega\mu}$$

Sovrapposizione di due onde polarizzate linearmente: onda risultante polarizzata ellitticamente

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1)

$$\mathbf{E}_{0R} = E_{0R}\mathbf{e}_0$$
$$\mathbf{H}_{0R} = H_{0R}\mathbf{h}_0$$
$$\boldsymbol{\beta} = \beta\,\boldsymbol{\beta}_0$$

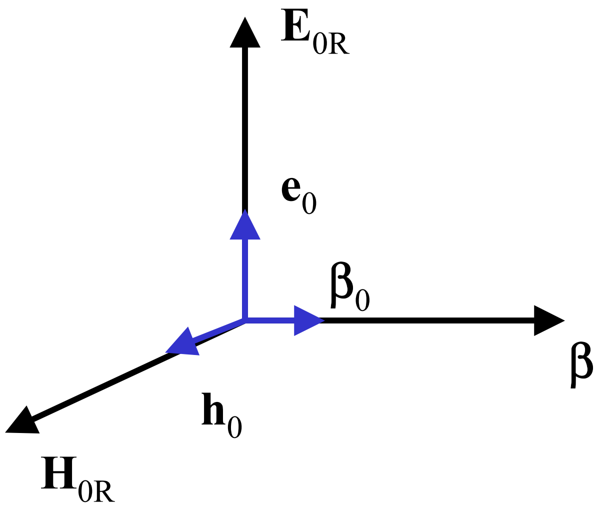
$$\boxed{\begin{gathered}\boldsymbol{\beta}\cdot\mathbf{E}_{0R} = 0 \\ \mathbf{H}_{0R} = \frac{\boldsymbol{\beta}\times\mathbf{E}_{0R}}{\omega\mu}\end{gathered}}$$

$$\beta\,\boldsymbol{\beta}_0\cdot E_{0R}\mathbf{e}_0 = 0 \Rightarrow \boldsymbol{\beta}_0\cdot\mathbf{e}_0 = 0$$

$$\mathbf{H}_{0R} = H_{0R}\mathbf{h}_0 = \frac{\beta\,\boldsymbol{\beta}_0\times E_{0R}\mathbf{e}_0}{\omega\mu}$$

$$\mathbf{h}_0 = \boldsymbol{\beta}_0\times\mathbf{e}_0$$

$$H_{0R} = \frac{\beta\,E_{0R}}{\omega\mu} = \frac{\omega\sqrt{\mu\varepsilon}\,E_{0R}}{\omega\mu} = \sqrt{\frac{\varepsilon}{\mu}}\,E_{0R}$$



$$\frac{E_{0R}}{H_{0R}} = \varsigma = \sqrt{\frac{\mu}{\varepsilon}}$$ *Impedenza caratteristica del mezzo*

$$\varsigma_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi \quad \Omega = 377 \quad \Omega$$ *vuoto*

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Vettore di Poynting per l'onda piana uniforme:

$$\mathbf{\Pi} = \frac{1}{2}\mathbf{E}\times\mathbf{H}^* = \frac{1}{2}\mathbf{E}_{0R}e^{-j\boldsymbol{\beta}\cdot\mathbf{r}}\times\mathbf{H}_{0R}{}^{*}e^{j\boldsymbol{\beta}\cdot\mathbf{r}} = \frac{1}{2}E_{0R}\mathbf{e}_0\times H_{0R}{}^{*}\mathbf{h}_0 = \frac{1}{2}\varsigma\, H_{0R}H_{0R}{}^{*}\boldsymbol{\beta}_0$$

$$\mathbf{e}_0\times\mathbf{h}_0 = \mathbf{e}_0\times\left(\boldsymbol{\beta}_0\times\mathbf{e}_0\right) = \boldsymbol{\beta}_0$$

Vettore costante reale diretto come il vettore di propagazione

Stesso procedimento vale per la soluzione 2)

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1) Mezzo non dispersivo e non dissipativo

ε, μ *reali e positive*; $\sigma = 0$

$\alpha \perp \beta$ ***Onda piana non uniforme***

Consideriamo il campo elettrico polarizzato linearmente

$\mathbf{E_0} = \mathbf{E}_{0R} + j\mathbf{E}_{0J}$ *ad esempio* $E_{0J}=0$

$\boldsymbol{\beta} = \beta\, \mathbf{z}_0$

$\boldsymbol{\alpha} = \alpha\, \mathbf{y}_0$

$$\mathbf{k} \cdot \mathbf{E_0} = 0 \longrightarrow (\boldsymbol{\beta} - j\boldsymbol{\alpha}) \cdot E_{0R}\mathbf{e}_0 = 0 \longrightarrow \begin{matrix} \boldsymbol{\beta} \cdot \mathbf{e}_0 = 0 \\ \boldsymbol{\alpha} \cdot \mathbf{e}_0 = 0 \end{matrix}$$

$$\mathbf{E_0} = \mathbf{E}_{0R} = E_{0R}\mathbf{e_0} = E_{0R}\mathbf{x}_0$$

$$\mathbf{E} = \mathbf{E}(y,z) = E_{0R}\mathbf{x}_0 e^{-\alpha y} e^{-j\beta \mathbf{z}}$$

$$\mathbf{H} = \mathbf{H}(y,z) = \frac{\mathbf{k} \times \mathbf{E}_0}{\omega\mu} e^{-j(\boldsymbol{\beta} - j\boldsymbol{\alpha})\cdot \mathbf{r}} = \left(\frac{\beta E_{0R}}{\omega\mu} \mathbf{z}_0 \times \mathbf{x}_0 - j \frac{\alpha E_{0R}}{\omega\mu} \mathbf{y}_0 \times \mathbf{x}_0 \right) e^{-\alpha y} e^{-j\beta z} =$$

$$= \frac{E_{0R}}{\omega\mu} (\beta\, \mathbf{y}_0 + j\alpha\, \mathbf{z}_0) e^{-\alpha y} e^{-j\beta z}$$

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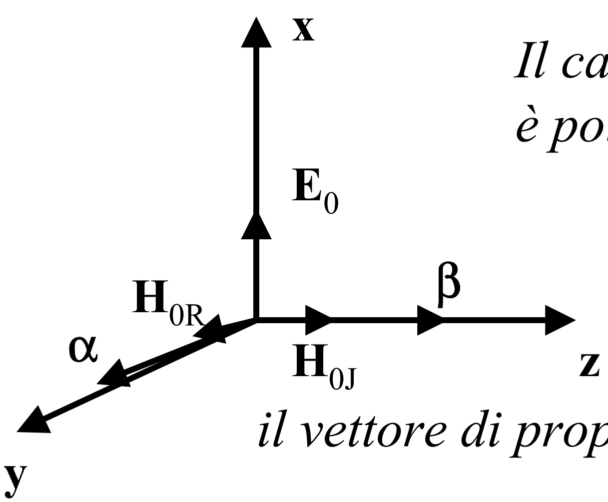


Il campo elettrico è polarizzato linearmente, quello magnetico è polarizzato ellitticamente

il vettore di propagazione $\underline{k}=\underline{\beta}-j\underline{\alpha}$ è ortogonale al campo elettrico

Onda TE
(TrasversoElettrica)

Vettore di Poynting per l'onda piana non uniforme attenuata:

$$\mathbf{\Pi}=\frac{1}{2}\mathbf{E}\times\mathbf{H}^*=\frac{1}{2}\mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}\times\mathbf{H}_0^{\,*}e^{j\mathbf{k}^*\cdot\mathbf{r}}=\frac{1}{2}\mathbf{E}_0\times\frac{\mathbf{k}^*\times\mathbf{E}_0^{\,*}}{\omega\mu}e^{-j\left(\mathbf{k}-\mathbf{k}^*\right)\cdot\mathbf{r}}=$$

$$=\frac{1}{2\omega\mu}\left[\left(\mathbf{E}_0\cdot\mathbf{E}_0^{\,*}\right)\mathbf{k}^*-\left(\mathbf{E}_0\cdot\mathbf{k}^*\right)\mathbf{E}_0^{\,*}\right]e^{-2\boldsymbol{\alpha}\cdot\mathbf{r}}=\frac{E_0^{\,2}}{2\omega\mu}e^{-2\boldsymbol{\alpha}\cdot\mathbf{r}}\mathbf{k}^*=\frac{E_0^{\,2}}{2\omega\mu}e^{-2\boldsymbol{\alpha}\cdot\mathbf{r}}\left(\boldsymbol{\beta}+j\boldsymbol{\alpha}\right)$$

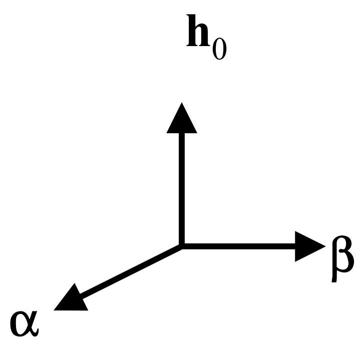
reale, diretta come β

immaginaria, diretta come α

$$\Pi = \frac{{E_{0R}}^2}{2\omega\mu} e^{-2\alpha\, y} \left(\beta\, \mathbf{z_0} + j\alpha\, \mathbf{y_0} \right)$$

Se si parte dal considerare il campo magnetico polarizzato linearmente si arriva in modo duale ad un'onda

Onda TM
(TrasversoMagnetica)



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2) Mezzo non dispersivo e dissipativo ε, μ *reali e positive*; $\sigma \neq 0$

$\boldsymbol{\beta}$ e $\boldsymbol{\alpha}$ *entrambi non nulli e non perpendicolari*

Caso particolare $\alpha // \beta$ ***Onda piana uniforme e attenuata***

$$\mathbf{k} = (\beta - j\alpha)\boldsymbol{\beta}_0 = k\,\boldsymbol{\beta}_0 = \omega\sqrt{\mu\varepsilon_c}\,\boldsymbol{\beta}_0$$

$$\mathbf{k} \cdot \mathbf{E}_0 = 0 \quad \Rightarrow \quad \boldsymbol{\beta}_0 \cdot \mathbf{E}_0 = 0$$

$$\mathbf{H}_0 = \frac{k}{\omega\mu}\boldsymbol{\beta}_0 \times \mathbf{E}_0$$

$\Downarrow$

Onda TEM

$$\mathbf{E_0} = E_0\mathbf{e_0}$$ *campo elettrico polarizzato linearmente*

$$\mathbf{H_0} = \frac{kE_0}{\omega\mu}\boldsymbol{\beta_0}\times\mathbf{e}_0 = H_0\mathbf{h_0}$$ *anche il campo magnetico è polarizzato linearmente*

$$\boxed{\begin{array}{ll} \mathbf{h_0} = \boldsymbol{\beta_0}\times\mathbf{e}_0 & H_0 = \dfrac{kE_0}{\omega\mu} = \dfrac{\omega\sqrt{\mu\varepsilon_c}}{\omega\mu}E_0 = \sqrt{\dfrac{\varepsilon_c}{\mu}}E_0 \\[2ex] \dfrac{E_0}{H_0} = \sqrt{\dfrac{\mu}{\varepsilon_c}} = \varsigma \longleftarrow & \textit{quantità complessa} \end{array}}$$

Vettore di Poynting per l'onda piana uniforme attenuata:

$$\mathbf{\Pi} = \frac{1}{2}E_0\mathbf{e}_0 e^{-j\beta\,\boldsymbol{\beta}_0\cdot\mathbf{r}}e^{-\alpha\,\boldsymbol{\beta}_0\cdot\mathbf{r}}\times H_0^{\,*}\mathbf{h}_0 e^{j\beta\,\boldsymbol{\beta}_0\cdot\mathbf{r}}e^{-\alpha\,\boldsymbol{\beta}_0\cdot\mathbf{r}} = \frac{1}{2}\varsigma\,\underline{H_0H_0^{\,*}e^{-2\alpha\,\boldsymbol{\beta}_0\cdot\mathbf{r}}}\;\boldsymbol{\beta}_0$$

complessa, diretta come β, che si attenua con cost 2α nella stessa direzione

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Costanti secondarie del mezzo

Costanti **primarie** *del mezzo:* ε, μ, σ

Costanti **secondarie** *del mezzo:* k, ζ

$$k = \omega\sqrt{\mu\varepsilon_c} = k_R - jk_J$$

$$k_R = \beta$$

$$k_J = \alpha$$

$$k_R = \omega\sqrt{\frac{|\varepsilon_c\mu| + \mathrm{Re}(\varepsilon_c\mu)}{2}}$$

$$k_J = \omega\sqrt{\frac{|\varepsilon_c\mu| - \mathrm{Re}(\varepsilon_c\mu)}{2}}$$

$$\varsigma = \sqrt{\frac{\mu}{\varepsilon_c}} = \frac{\omega\mu}{k} = \varsigma_R + j\varsigma_J$$

$$\varsigma_R = \frac{\omega\mu\ k_R}{k_R^{\,2} + k_J^{\,2}}$$

$$\varsigma_J = \frac{\omega\mu\ k_J}{k_R^{\,2} + k_J^{\,2}}$$

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Costanti secondarie del mezzo

Trattazione analitica

$$k = k_R - jk_J$$

$$k^2 = k_R^{\ 2} - k_J^{\ 2} - 2jk_R k_J = \omega^2 \mu\varepsilon - j\omega\mu\sigma$$

$$k_R^{\ 2} - k_J^{\ 2} = \omega^2 \mu\varepsilon$$

$$k_R k_J = \frac{\omega\mu\sigma}{2}$$

$$k_R = \omega\sqrt{\mu\varepsilon}\sqrt{\frac{1}{2}\left[1+\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}\right]} \qquad k_J = \omega\sqrt{\mu\varepsilon}\sqrt{\frac{1}{2}\left[-1+\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}\right]}$$

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Costanti secondarie del mezzo

$$\varsigma = \varsigma_R + j\varsigma_J$$

$$\varsigma^2 = \varsigma_R{}^2 - \varsigma_J{}^2 + 2j\varsigma_R\varsigma_J = \frac{\mu}{\varepsilon_c} = \frac{j\omega\mu}{\sigma + j\omega\varepsilon} = \frac{j\omega\mu(\sigma - j\omega\varepsilon)}{\sigma^2 + \omega^2\varepsilon^2}$$

$$\varsigma_R{}^2 - \varsigma_J{}^2 = \frac{\omega^2\mu\varepsilon}{\sigma^2 + \omega^2\varepsilon^2}$$

$$\varsigma_R\varsigma_J = \frac{1}{2}\frac{\omega\mu\sigma}{\sigma^2 + \omega^2\varepsilon^2}$$

$$\varsigma_R = \frac{k_R}{\sqrt{\sigma^2 + \omega^2\varepsilon^2}} \qquad \varsigma_J = \frac{k_J}{\sqrt{\sigma^2 + \omega^2\varepsilon^2}}$$

Esempi

Onda piana in un buon dielettrico

$$\sigma << \omega\varepsilon$$

$$k_R \cong \omega\sqrt{\mu\varepsilon} \qquad k_R k_J = \frac{1}{2}\omega\mu\sigma$$

$$k_J = \frac{\omega\mu\sigma}{2k_R} \cong \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

$$\varsigma_R \cong \frac{k_R}{\omega\,\varepsilon} \cong \sqrt{\frac{\mu}{\varepsilon}}$$

$$\varsigma_J \cong \frac{k_J}{\omega\,\varepsilon} \cong \frac{\sigma}{2\omega\varepsilon}\sqrt{\frac{\mu}{\varepsilon}}$$

Onda piana in un buon conduttore

$$\sigma >> \omega\varepsilon$$

$$\beta = k_R \cong \sqrt{\frac{\omega\mu\sigma}{2}} \qquad k_R k_J = \frac{1}{2}\omega\mu\sigma$$

$$\alpha = k_J = \frac{\omega\mu\sigma}{2k_R} \cong \sqrt{\frac{\omega\mu\sigma}{2}}$$

$$\varsigma_R \cong \frac{k_R}{\sigma} \cong \sqrt{\frac{\omega\mu}{2\sigma}}$$

$$\varsigma_J \cong \frac{k_J}{\sigma} \cong \sqrt{\frac{\omega\mu}{2\sigma}}$$

$$k = (1 - j)\sqrt{\frac{\omega\mu\sigma}{2}} \Rightarrow \delta = \frac{1}{k_J} = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}}$$ *Profondità di penetrazione*

$$\varsigma = (1 + j)\sqrt{\frac{\omega\mu}{2\sigma}} = (1 + j)\sigma\delta = (1 + j)R_s \qquad \delta = \frac{1}{k_R} = \frac{1}{\beta}$$

$$\mathbf{E_0} = E_0\mathbf{e_0} \rightarrow \mathbf{E_0} = E_0\mathbf{x_0} \qquad \boldsymbol{\beta}_\mathbf{0} = \beta_0\mathbf{z_0}$$

$$\mathbf{H_0} = \frac{kE_0}{\omega\mu}\boldsymbol{\beta}_\mathbf{0} \times \mathbf{e}_0 = H_0\mathbf{h_0} \rightarrow \mathbf{H_0} = H_0\mathbf{y_0}$$

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}(z) = E_x(z)\mathbf{x_0} = E_0 e^{-jkz}\mathbf{x_0} = E_0 e^{-j\beta z}e^{-\alpha z}\mathbf{x_0}$$

$$\mathbf{H}(\mathbf{r}) = \mathbf{H}(z) = H_y(z)\mathbf{y_0} = H_0 e^{-jkz}\mathbf{y_0} = H_0 e^{-j\beta z}e^{-\alpha z}\mathbf{y_0}$$

$$\mathbf{\Pi}(\mathbf{r}) = \mathbf{\Pi}(z) = \frac{1}{2}\varsigma H_0 H_0^* e^{-2\alpha z}\mathbf{z}_0$$

Es: rame

Per f=1 MHz,

$\sigma = 5.8 \quad 10^7 \quad \Omega^{-1} \quad m^{-1}$

$\mu = \mu_0$

$k_R \cong k_J \cong 15132 \ m^{-1}$

$$\delta = \frac{1}{k_J} = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}} = 0.065 \quad 10^{-3}\,m$$

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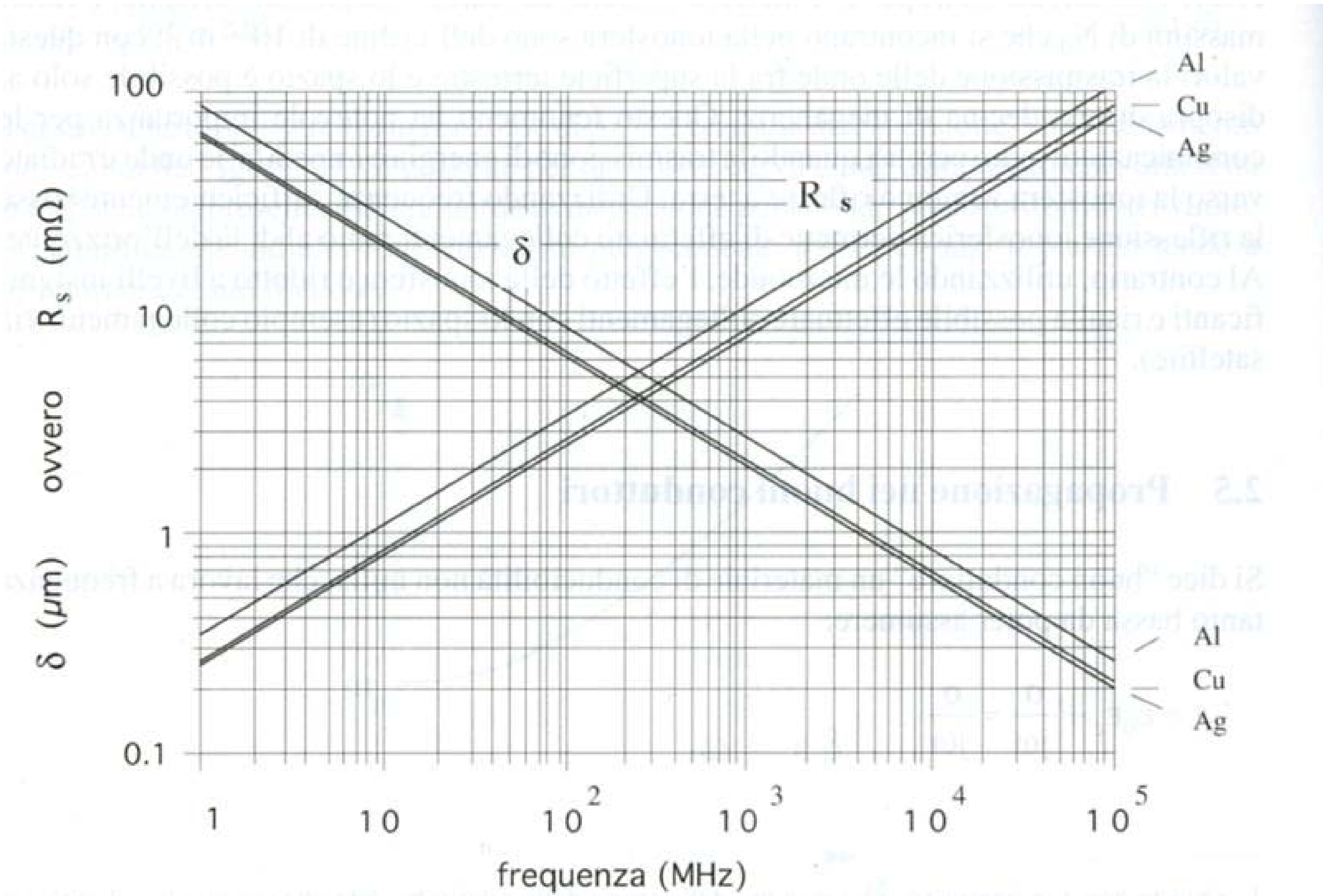
Esempio onda piana in un buon conduttore

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

$$R_s = \frac{1}{\delta\sigma}$$



Esempio onda piana in un buon conduttore

Calcolare la profondità di penetrazione di alluminio, rame, oro e argento alla frequenza di 10 GHz:

$$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{1}{\pi f \mu_0 \sigma}}\sqrt{\frac{1}{\sigma}} = \sqrt{\frac{1}{\pi\left(10^{10}\right)\left(4\pi 10^{-7}\right)}}\sqrt{\frac{1}{\sigma}} = 5\,10^{-3}\sqrt{\frac{1}{\sigma}}$$

alluminio: $\Rightarrow 5\,10^{-3}\sqrt{\frac{1}{3.8\,10^{7}}} = 8.1\ 10^{-7} \quad m$

rame: $\Rightarrow 5\,10^{-3}\sqrt{\frac{1}{5.8\,10^{7}}} = 6.6\ 10^{-7} \quad m$

oro: $\Rightarrow 5\,10^{-3}\sqrt{\frac{1}{4.1\,10^{7}}} = 7.86\ 10^{-7} \quad m$

argento: $\Rightarrow 5\,10^{-3}\sqrt{\frac{1}{6.1\,10^{7}}} = 6.4\ 10^{-7} \quad m$

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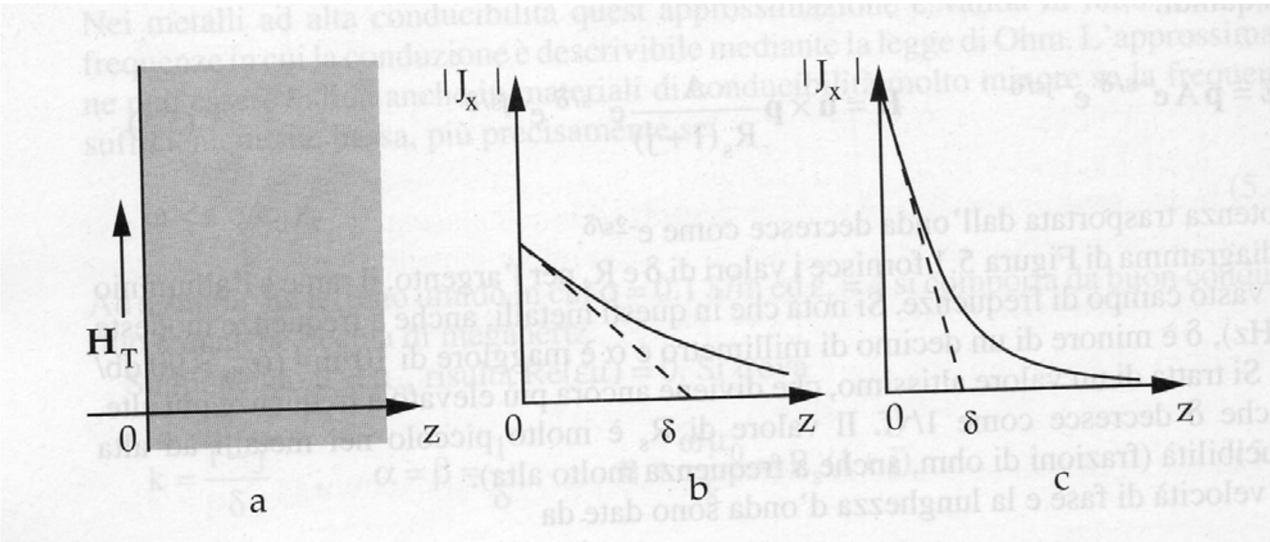
Propagazione di un'onda piana in un buon conduttore



$$\mathbf{k} = (\beta - j\alpha)\mathbf{z_0} = k\,\mathbf{z_0}$$

$$\mathbf{k}\cdot\mathbf{e}_0 = \mathbf{z_0}\cdot\mathbf{e}_0 = 0$$

$$\varsigma = (1+j)R_s$$

$$\mathbf{E} = E_0\mathbf{e_0}e^{-\frac{z}{\delta}}e^{-j\frac{z}{\delta}}$$

$$\mathbf{H} = \mathbf{H}_0 e^{-\frac{z}{\delta}}e^{-j\frac{z}{\delta}} = \frac{kE_0}{\omega\mu}\mathbf{z_0}\times\mathbf{e_0}e^{-\frac{z}{\delta}}e^{-j\frac{z}{\delta}} = \frac{E_0}{\varsigma}\mathbf{z_0}\times\mathbf{e_0}e^{-\frac{z}{\delta}}e^{-j\frac{z}{\delta}} = \frac{E_0}{(1+j)R_s}\mathbf{z_0}\times\mathbf{e_0}e^{-\frac{z}{\delta}}e^{-j\frac{z}{\delta}}$$

in z=0

$$\mathbf{z_0}\times\mathbf{e_0}\times\mathbf{z_0} = \mathbf{e_0}$$

$$\frac{E_0}{(1+j)R_s}\mathbf{z_0}\times\mathbf{e_0} = \mathbf{H}_T \;\Rightarrow\; \frac{E_0}{(1+j)R_s}\mathbf{e_0} = \mathbf{H}_T\times\mathbf{z_0} \;\Rightarrow\; \begin{cases} \mathbf{E} = (1+j)R_s\mathbf{H}_T\times\mathbf{z_0}e^{-\frac{z}{\delta}}e^{-j\frac{z}{\delta}} \\ \mathbf{H} = \mathbf{H}_T e^{-\frac{z}{\delta}}e^{-j\frac{z}{\delta}} \end{cases}$$

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la densità di corrente:

Effetto pelle

$$\mathbf{J} = \sigma\mathbf{E} = \frac{(1+j)}{\delta}\mathbf{H}_T \times \mathbf{z_0} e^{-\frac{z}{\delta}} e^{-j\frac{z}{\delta}}$$

è diretta parallelamente alla superficie ed ha valori sensibili solo in uno strato superficiale di spessore δ

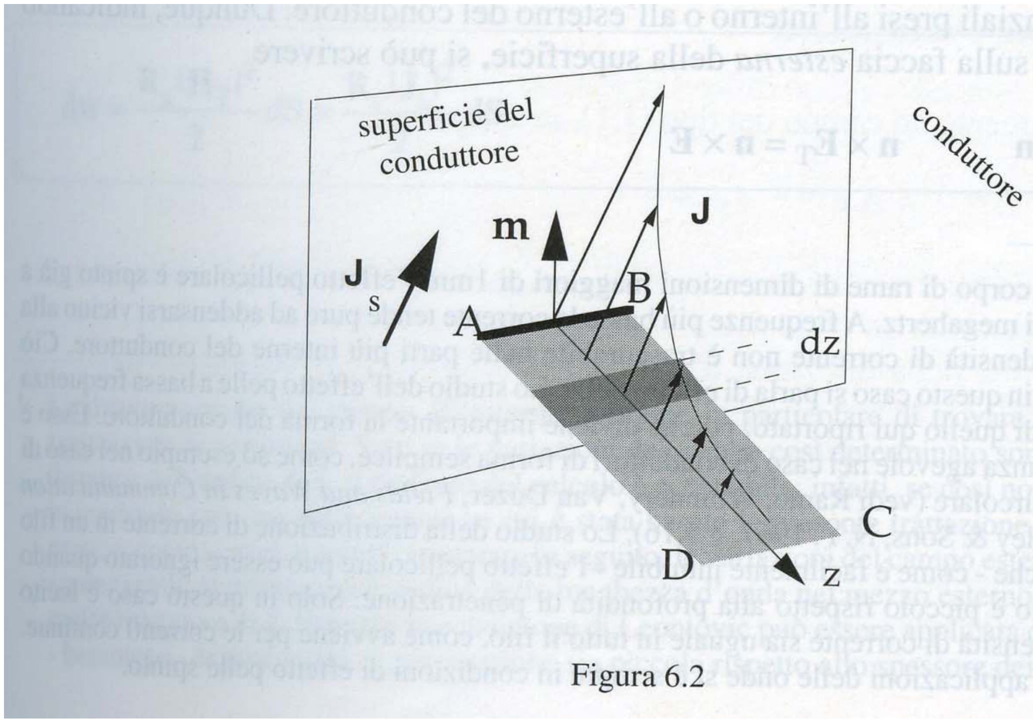


Figura 6.2

$$\mathbf{m} \cdot \mathbf{J_s} L = L\int_0^\infty \mathbf{m} \cdot \mathbf{J}\, dz =$$

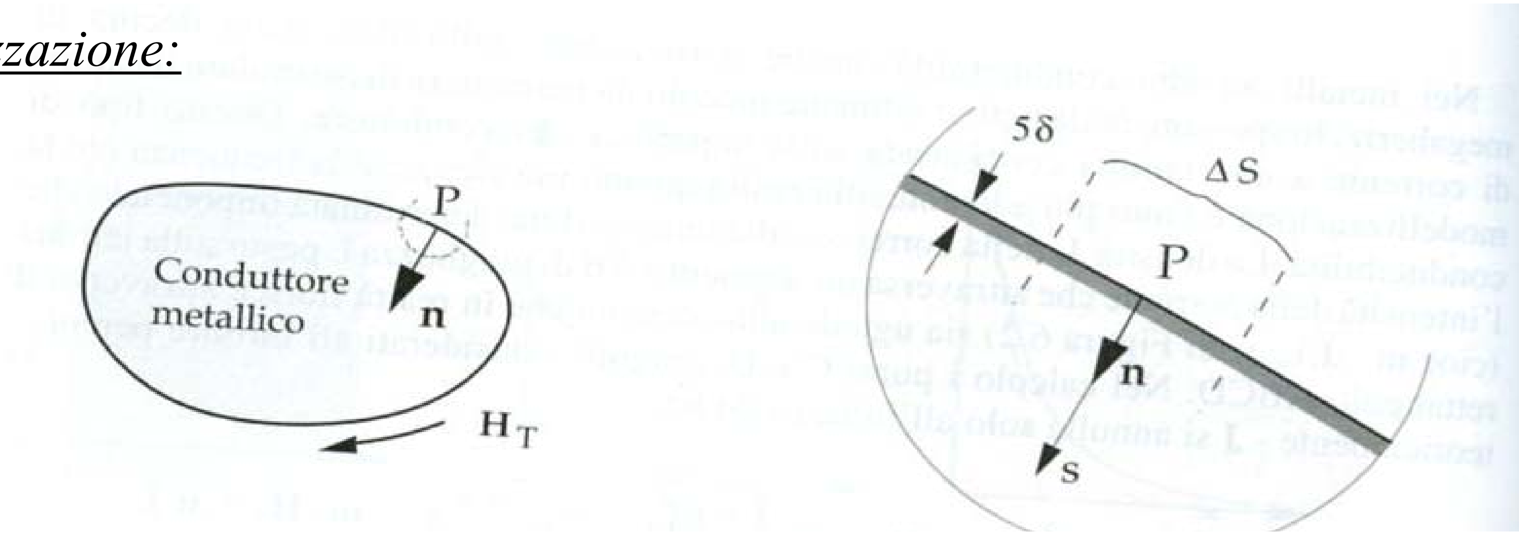
$$= L\mathbf{m} \cdot \mathbf{H}_T \times \mathbf{z_0} \frac{(1+j)}{\delta} \int_0^\infty e^{-\frac{z}{\delta}} e^{-j\frac{z}{\delta}}\, dz =$$

$$= \mathbf{m} \cdot \mathbf{H}_T \times \mathbf{z_0} L$$

$$\mathbf{J_s} = \mathbf{H}_T \times \mathbf{z_0}$$

Nei metalli ad alta conducibilità lo spessore della 'pelle' è talmente piccolo da poter assimilare il campo di corrente ad una ***lamina*** *concentrata sulla superficie del conduttore*

Generalizzazione:



Ipotesi:

- *lo spessore della 'pelle' molto minore di tutte le dimensioni caratteristiche del corpo*
- *lo spessore della 'pelle' molto minore delle minime distanze per cui si hanno apprezzabili variazioni di H_T sulla superficie*
- *Δs molto maggiore di δ ma piccolo a sufficienza da poter considerare l'elemento piano*

impedenza superficiale

$$\boxed{\begin{aligned} \mathbf{n}\times\mathbf{E} &= \mathbf{H}_T(1+j)R_s = \varsigma\,\mathbf{H}_T \\ \mathbf{J}_s &= \mathbf{H}\times\mathbf{n} \end{aligned}}$$

superficie o parete d'impedenza

$$\mathbf{n}\times\mathbf{E} = 0$$

parete elettrica

Condizione di Leontovic

$$\mathbf{W} = \frac{R_S}{2}\int_{\mathbf{S}}\left|\mathbf{H_T}\right|^2 dS = \frac{R_S}{2}\int_{\mathbf{S}}\left|\mathbf{J_S}\right|^2 dS$$ *potenza dissipata in un corpo conduttore*

$$\boldsymbol{\mathcal{P}}(z) = \frac{1}{2}\mathrm{Re}\big[\boldsymbol{E}(z)\times\boldsymbol{H}^*(z)\big] = \frac{1}{2}\mathrm{Re}\left[\frac{1}{\eta_c^*}\boldsymbol{E}_0\times(\hat{\mathbf{z}}\times\boldsymbol{E}_0^*)e^{-(\alpha+j\beta)z}e^{-(\alpha-j\beta)z}\right]$$

$$= \hat{\mathbf{z}}\,\frac{1}{2}\mathrm{Re}(\eta_c^{-1})\,|\boldsymbol{E}_0|^2 e^{-2\alpha z} = \hat{\mathbf{z}}\,\mathcal{P}(0)e^{-2\alpha z} = \hat{\mathbf{z}}\,\mathcal{P}(z)$$

Thus, the power per unit area flowing past the point z in the forward z-direction will be:

$$\boxed{\mathcal{P}(z) = \mathcal{P}(0)e^{-2\alpha z}} \tag{2.6.16}$$

The quantity $\mathcal{P}(0)$ is the power per unit area flowing past the point $z = 0$. Denoting the real and imaginary parts of η_c by η', η'', so that, $\eta_c = \eta' + j\eta''$, and noting that $|\boldsymbol{E}_0| = |\eta_c\boldsymbol{H}_0\times\hat{\mathbf{z}}| = |\eta_c||\boldsymbol{H}_0|$, we may express $\mathcal{P}(0)$ in the equivalent forms:

$$\mathcal{P}(0) = \frac{1}{2}\mathrm{Re}(\eta_c^{-1})\,|\boldsymbol{E}_0|^2 = \frac{1}{2}\eta'\,|\boldsymbol{H}_0|^2 \tag{2.6.17}$$

The attenuation coefficient α is measured in *nepers per meter*. However, a more practical way of expressing the power attenuation is in dB per meter. Taking logs of Eq. (2.6.16), we have for the dB attenuation at z, relative to $z = 0$:

$$A_{\mathrm{dB}}(z) = -10\log_{10}\left[\frac{\mathcal{P}(z)}{\mathcal{P}(0)}\right] = 20\log_{10}(e)\,\alpha z = 8.686\,\alpha z \tag{2.6.18}$$

where we used the numerical value $20\log_{10}e = 8.686$. Thus, the quantity $\alpha_{\mathrm{dB}} = 8.686\,\alpha$ is the attenuation in *dB per meter*:

$$\alpha_{\mathrm{dB}} = 8.686\,\alpha \qquad (\mathrm{dB/m}) \tag{2.6.19}$$

Another way of expressing the power attenuation is by means of the so-called *penetration* or *skin depth* defined as the inverse of α:

$$\boxed{\delta = \frac{1}{\alpha}} \qquad \text{(skin depth)} \tag{2.6.20}$$

Then, Eq. (2.6.18) can be rewritten in the form:

$$A_{\mathrm{dB}}(z) = 8.686\,\frac{z}{\delta} \qquad \text{(attenuation in dB)} \tag{2.6.21}$$

This gives rise to the so-called "9-dB per delta" rule, that is, every time z is increased by a distance δ, the attenuation increases by $8.686 \simeq 9$ dB.

A useful way to represent Eq. (2.6.16) in practice is to consider its infinitesimal version obtained by differentiating it with respect to z and solving for α:

$$\mathcal{P}'(z) = -2\alpha\mathcal{P}(0)e^{-2\alpha z} = 2\alpha\mathcal{P}(z) \quad \Rightarrow \quad \alpha = -\frac{\mathcal{P}'(z)}{2\mathcal{P}(z)}$$

The quantity $\mathcal{P}'_{\text{loss}} = -\mathcal{P}'$ represents the power lost from the wave per unit length (in the propagation direction.) Thus, the attenuation coefficient is the ratio of the power loss per unit length to twice the power transmitted:

$$\boxed{\alpha = \frac{P'_{\text{loss}}}{2P_{\text{transm}}}} \qquad \text{(attenuation coefficient)} \qquad (2.6.22)$$

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If there are several physical mechanisms for the power loss, then α becomes the *sum* over all possible cases. For example, in a waveguide or a coaxial cable filled with a slightly lossy dielectric, power will be lost because of the small conduction/polarization currents set up within the dielectric and also because of the ohmic losses in the walls of the guiding conductors, so that the total α will be $\alpha = \alpha_{\text{diel}} + \alpha_{\text{walls}}$.

Next, we verify that the exponential loss of power from the propagating wave is due to ohmic heat losses. In Fig. 2.6.1, we consider a volume $dV = l\,dA$ of area dA and length l along the z-direction.

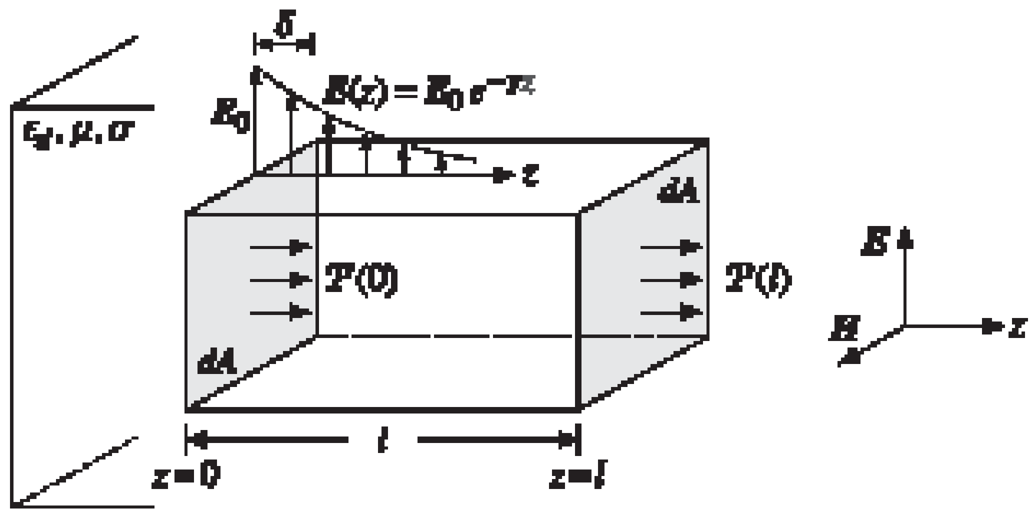

Fig. 2.6.1 Power flow in lossy dielectric.

From the definition of $\mathcal{P}(z)$ as power flow per unit area, it follows that the power entering the area dA at $z=0$ will be $dP_{\text{in}} = \mathcal{P}(0)\,dA$, and the power leaving that area at $z=l$, $dP_{\text{out}} = \mathcal{P}(l)\,dA$. The difference $dP_{\text{loss}} = dP_{\text{in}} - dP_{\text{out}} = [\mathcal{P}(0) - \mathcal{P}(l)]\,dA$ will be the power lost from the wave within the volume $l\,dA$. Because $\mathcal{P}(l) = \mathcal{P}(0)\,e^{-2\alpha l}$, we have for the power loss per unit area:

$$\frac{dP_{\text{loss}}}{dA} = \mathcal{P}(0) - \mathcal{P}(l) = \mathcal{P}(0)\left(1 - e^{-2\alpha l}\right) = \frac{1}{2}\,\text{Re}\left(\eta_c^{-1}\right)\,|E_0|^2\left(1 - e^{-2\alpha l}\right) \qquad (2.6.23)$$

On the other hand, according to Eq. (2.6.3), the ohmic power loss per unit volume will be $\omega\epsilon''|E(z)|^2/2$. Integrating this quantity from $z = 0$ to $z = l$ will give the total ohmic losses within the volume $l\,dA$ of Fig. 2.6.1. Thus, we have:

$$dP_{\text{ohmic}} = \frac{1}{2}\omega\epsilon''\int_0^l |E(z)|^2\,dz\,dA = \frac{1}{2}\omega\epsilon''\left[\int_0^l |E_0|^2 e^{-2\alpha z}\,dz\right]dA\,, \qquad \text{or,}$$

$$\frac{dP_{\text{ohmic}}}{dA} = \frac{\omega\epsilon''}{4\alpha}|E_0|^2(1-e^{-2\alpha l}) \tag{2.6.24}$$

Are the two expressions in Eqs. (2.6.23) and (2.6.24) equal? The answer is yes, as follows from the relationship among the quantities $\eta_c, \epsilon'', \alpha$ (see Problem 2.12):

$$\boxed{\mathrm{Re}(\eta_c^{-1}) = \frac{\omega\epsilon''}{2\alpha}} \tag{2.6.25}$$

Thus, the power lost from the wave is entirely accounted for by the ohmic losses within the propagation medium. The equality of (2.6.23) and (2.6.24) is an example of the more general relationship proved in Problem 1.5.

In the limit $l \to \infty$, we have $\mathcal{P}(l) \to 0$, so that $dP_{\text{ohmic}}/dA = \mathcal{P}(0)$, which states that all the power that enters at $z = 0$ will be dissipated into heat inside the semi-infinite medium. Using Eq. (2.6.17), we summarize this case:

$$\boxed{\frac{dP_{\text{ohmic}}}{dA} = \frac{1}{2}\mathrm{Re}(\eta_c^{-1})\,|E_0|^2 = \frac{1}{2}\eta'\,|H_0|^2} \qquad \text{(ohmic losses)} \tag{2.6.26}$$

This result will be used later on to calculate ohmic losses of waves incident on lossy dielectric or conductor surfaces, as well as conductor losses in waveguide and transmission line problems.

Example 2.6.1: The absorption coefficient α of water reaches a minimum over the visible spectrum—a fact undoubtedly responsible for why the visible spectrum is visible.

Recent measurements [116] of the absorption coefficient show that it starts at about 0.01 nepers/m at 380 nm (violet), decreases to a minimum value of 0.0044 nepers/m at 418 nm (blue), and then increases steadily reaching the value of 0.5 nepers/m at 600 nm (red). Determine the penetration depth δ in meters, for each of the three wavelengths.

Determine the depth in meters at which the light intensity has decreased to 1/10th its value at the surface of the water. Repeat, if the intensity is decreased to 1/100th its value.

Solution: The penetration depths $\delta = 1/\alpha$ are:

$$\delta = 100,\ 227.3,\ 2\text{ m} \quad \text{for} \quad \alpha = 0.01,\ 0.0044,\ 0.5\text{ nepers/m}$$

Using Eq. (2.6.21), we may solve for the depth $z = (A/8.9696)\delta$. Since a decrease of the light intensity (power) by a factor of 10 is equivalent to $A = 10$ dB, we find $z = (10/8.9696)\delta = 1.128\,\delta$, which gives: $z = 112.8,\ 256.3,\ 2.3$ m. A decrease by a factor of $100 = 10^{20/10}$ corresponds to $A = 20$ dB, effectively doubling the above depths. □

Example 2.6.2: A microwave oven operating at 2.45 GHz is used to defrost a frozen food having complex permittivity $\epsilon_c = (4 - j)\epsilon_0$ farad/m. Determine the strength of the electric field at a depth of 1 cm and express it in dB and as a percentage of its value at the surface. Repeat if $\epsilon_c = (45 - 15j)\epsilon_0$ farad/m.

Solution: The free-space wavenumber is $k_0 = \omega\sqrt{\mu_0\epsilon_0} = 2\pi f/c_0 = 2\pi(2.45\times10^9)/(3\times10^8) =$ 51.31 rad/m. Using $k_c = \omega\sqrt{\mu_0\epsilon_c} = k_0\sqrt{\epsilon_c/\epsilon_0}$, we calculate the wavenumbers:

$$
\begin{aligned}
k_c &= \beta - j\alpha = 51.31\sqrt{4-j} = 51.31\,(2.02 - 0.25j) = 103.41 - 12.73j\ \text{m}^{-1} \\
k_c &= \beta - j\alpha = 51.31\sqrt{45-15j} = 51.31\,(6.80 - 1.10j) = 348.84 - 56.61j\ \text{m}^{-1}
\end{aligned}
$$

The corresponding attenuation constants and penetration depths are:

$$
\begin{aligned}
\alpha &= 12.73\ \text{nepers/m}, &\quad \delta &= 7.86\ \text{cm} \\
\alpha &= 56.61\ \text{nepers/m}, &\quad \delta &= 1.77\ \text{cm}
\end{aligned}
$$

It follows that the attenuations at 1 cm will be in dB and in absolute units:

$$
\begin{aligned}
A &= 8.686\,z/\delta = 1.1\ \text{dB}, &\quad 10^{-A/20} &= 0.88 \\
A &= 8.686\,z/\delta = 4.9\ \text{dB}, &\quad 10^{-A/20} &= 0.57
\end{aligned}
$$

Thus, the fields at a depth of 1 cm are 88% and 57% of their values at the surface. The complex permittivities of some foods may be found in [117]. □

A convenient way to characterize the degree of ohmic losses is by means of the *loss tangent*, originally defined in Eq. (1.9.28). Here, we set:

$$\tau = \tan\theta = \frac{\epsilon''}{\epsilon'} = \frac{\sigma + \omega\epsilon''_d}{\omega\epsilon'_d} \tag{2.6.27}$$

Then, $\epsilon_c = \epsilon' - j\epsilon'' = \epsilon'(1 - j\tau) = \epsilon'_d(1 - j\tau)$. Therefore, k_c, η_c may be written as:

$$k_c = \omega\sqrt{\mu\epsilon'_d}\,(1 - j\tau)^{1/2}\,, \quad \eta_c = \sqrt{\frac{\mu}{\epsilon'_d}}\,(1 - j\tau)^{-1/2} \tag{2.6.28}$$

The quantities $c_d = 1/\sqrt{\mu\epsilon'_d}$ and $\eta_d = \sqrt{\mu/\epsilon'_d}$ would be the speed of light and characteristic impedance of an equivalent lossless dielectric with permittivity ϵ'_d.

In terms of the loss tangent, we may characterize *weakly* lossy media versus *strongly* lossy ones by the conditions $\tau \ll 1$ versus $\tau \gg 1$, respectively. These conditions depend on the operating frequency ω:

$$\frac{\sigma + \omega\epsilon''_d}{\omega\epsilon'_d} \ll 1 \quad \text{versus} \quad \frac{\sigma + \omega\epsilon''_d}{\omega\epsilon'_d} \gg 1$$

The expressions (2.6.28) may be simplified considerably in these two limits. Using the small-x Taylor series expansion $(1+x)^{1/2} \simeq 1 + x/2$, we find in the weakly lossy case $(1 - j\tau)^{1/2} \simeq 1 - j\tau/2$, and similarly, $(1 - j\tau)^{-1/2} \simeq 1 + j\tau/2$.

On the other hand, if $\tau \gg 1$, we may approximate $(1 - j\tau)^{1/2} \simeq (-j\tau)^{1/2} = e^{-j\pi/4}\tau^{1/2}$, where we wrote $(-j)^{1/2} = (e^{-j\pi/2})^{1/2} = e^{-j\pi/4}$. Similarly, $(1 - j\tau)^{-1/2} \simeq e^{j\pi/4}\tau^{-1/2}$. Thus, we summarize the two limits:

2.7 Propagation in Weakly Lossy Dielectrics

In the weakly lossy case, the propagation parameters k_c, η_c become:

$$
\begin{aligned}
k_c &= \beta - j\alpha = \omega\sqrt{\mu\epsilon_d'}\left(1 - j\frac{\tau}{2}\right) = \omega\sqrt{\mu\epsilon_d'}\left(1 - j\frac{\sigma + \omega\epsilon_d''}{2\omega\epsilon_d'}\right) \\
\eta_c &= \eta' + j\eta'' = \sqrt{\frac{\mu}{\epsilon_d'}}\left(1 + j\frac{\tau}{2}\right) = \sqrt{\frac{\mu}{\epsilon_d'}}\left(1 + j\frac{\sigma + \omega\epsilon_d''}{2\omega\epsilon_d'}\right)
\end{aligned}
\qquad (2.7.1)
$$

Thus, the phase and attenuation constants are:

$$
\beta = \omega\sqrt{\mu\epsilon_d'} = \frac{\omega}{c_d}\,, \quad \alpha = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon_d'}}(\sigma + \omega\epsilon_d'') = \frac{1}{2}\eta_d(\sigma + \omega\epsilon_d'') \qquad (2.7.2)
$$

For a slightly conducting dielectric with $\epsilon_d'' = 0$ and a small conductivity σ, Eq. (2.7.2) implies that the attenuation coefficient α is frequency-independent in this limit.

Example 2.7.1: Seawater has $\sigma = 4$ Siemens/m and $\epsilon_d = 81\epsilon_0$ (so that $\epsilon_d' = 81\epsilon_0$, $\epsilon_d'' = 0$.) Then, $n_d = \sqrt{\epsilon_d/\epsilon_0} = 9$, and $c_d = c_0/n_d = 0.33\times 10^8$ m/sec and $\eta_d = \eta_0/n_d = 377/9 = 41.89\ \Omega$. The attenuation coefficient (2.7.2) will be:

$$
\alpha = \frac{1}{2}\eta_d\sigma = \frac{1}{2}41.89\times 4 = 83.78 \text{ nepers/m} \quad\Rightarrow\quad \alpha_{\text{dB}} = 8.686\,\alpha = 728 \text{ dB/m}
$$

The corresponding skin depth is $\delta = 1/\alpha = 1.19$ cm. This result assumes that $\sigma \ll \omega\epsilon_d$, which can be written in the form $\omega \gg \sigma/\epsilon_d$, or $f \gg f_0$, where $f_0 = \sigma/(2\pi\epsilon_d)$. Here, we have $f_0 = 888$ MHz. For frequencies $f \lesssim f_0$, we must use the exact equations (2.6.28). For example, we find:

$$
\begin{array}{lll}
f = 1\text{ kHz}, & \alpha_{\text{dB}} = 1.09\text{ dB/m}, & \delta = 7.96\text{ m} \\
f = 1\text{ MHz}, & \alpha_{\text{dB}} = 34.49\text{ dB/m}, & \delta = 25.18\text{ cm} \\
f = 1\text{ GHz}, & \alpha_{\text{dB}} = 672.69\text{ dB/m}, & \delta = 1.29\text{ cm}
\end{array}
$$

Such extremely large attenuations explain why communication with submarines is impossible at high RF frequencies. □

2.8 Propagation in Good Conductors

A conductor is characterized by a large value of its conductivity σ, while its dielectric constant may be assumed to be real-valued $\epsilon_d = \epsilon$ (typically equal to ϵ_0.) Thus, its complex permittivity and loss tangent will be:

$$\epsilon_c = \epsilon - j\frac{\sigma}{\omega} = \epsilon\left(1 - j\frac{\sigma}{\omega\epsilon}\right), \quad \tau = \frac{\sigma}{\omega\epsilon} \tag{2.8.1}$$

A good conductor corresponds to the limit $\tau \gg 1$, or, $\sigma \gg \omega\epsilon$. Using the approximations of Eqs. (2.6.29) and (2.6.30), we find for the propagation parameters k_c, η_c:

$$\begin{aligned} k_c &= \beta - j\alpha = \omega\sqrt{\mu\epsilon}\sqrt{\frac{\tau}{2}}\,(1-j) = \sqrt{\frac{\omega\mu\sigma}{2}}\,(1-j) \\ \eta_c &= \eta' + j\eta'' = \sqrt{\frac{\mu}{\epsilon}}\sqrt{\frac{1}{2\tau}}\,(1+j) = \sqrt{\frac{\omega\mu}{2\sigma}}\,(1+j) \end{aligned} \tag{2.8.2}$$

Thus, the parameters β, α, δ are:

$$\boxed{\beta = \alpha = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma}}\,, \qquad \boxed{\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}} = \frac{1}{\sqrt{\pi f\mu\sigma}}} \tag{2.8.3}$$

where we replaced $\omega = 2\pi f$. The complex characteristic impedance η_c can be written in the form $\eta_c = R_s(1+j)$, where R_s is called the *surface resistance* and is given by the equivalent forms (where $\eta = \sqrt{\mu/\epsilon}$):

$$\boxed{R_s = \eta\sqrt{\frac{\omega\epsilon}{2\sigma}} = \sqrt{\frac{\omega\mu}{2\sigma}} = \frac{\alpha}{\sigma} = \frac{1}{\sigma\delta}} \tag{2.8.4}$$

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Example 2.8.1: For copper we have $\sigma = 5.8 \times 10^7$ Siemens/m. The skin depth at frequency f is:

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi \cdot 4\pi \cdot 10^{-7} \cdot 5.8 \cdot 10^7}} f^{-1/2} = 0.0661\, f^{-1/2} \qquad (f \text{ in Hz})$$

We find at frequencies of 1 kHz, 1 MHz, and 1 GHz:

$$\begin{aligned} f &= 1\,\text{kHz}, & \delta &= 2.09\ \text{mm} \\ f &= 1\,\text{MHz}, & \delta &= 0.07\ \text{mm} \\ f &= 1\,\text{GHz}, & \delta &= 2.09\ \mu\text{m} \end{aligned}$$

Thus, the skin depth is extremely small for good conductors at RF. □

Because δ is so small, the fields will attenuate rapidly within the conductor, depending on distance like $e^{-\gamma z} = e^{-\alpha z} e^{-j\beta z} = e^{-z/\delta} e^{-j\beta z}$. The factor $e^{-z/\delta}$ effectively confines the fields to within a distance δ from the surface of the conductor.

This allows us to define equivalent "surface" quantities, such as surface current and surface impedance. With reference to Fig. 2.6.1, we define the surface current density by integrating the density $\boldsymbol{J}(z) = \sigma \boldsymbol{E}(z) = \sigma \boldsymbol{E}_0 e^{-\gamma z}$ over the top-side of the volume $l\, dA$, and taking the limit $l \to \infty$: